THE B.I.E.M. APPLIED TO FLOW THROUGH POROUS MEDIA

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1.- THE PROBLEM.

This paper refers to the numerical solution of the classical DARCY'S problem of plane fluid through isotropic media.

As ti is well known if

$$h = z + \frac{P}{\gamma} \qquad (1)$$

is the piezometric height at a point of the medium under study the h potencial is an harmonic one and the mean flow can be asimilated to an irrotational perfect fluid one.

$$\nabla \cdot k \nabla h = \left\{ k = \text{cnt} \right\} = k \nabla^2 h = 0 \qquad (2)$$

The boundary conditions of the problem are:

impervious boundary

$$\underline{v} \cdot \underline{\nu} = 0 \Rightarrow k\, \partial h / \partial \nu = 0 \qquad (3)$$

porous boundary

$$h = H \qquad (4)$$

free boundary

$$\partial h / \partial \nu = 0 \Rightarrow k\, \partial z / \partial \nu = 0 \qquad (5)$$

If k is constant (homogeneous medium) the harmonic potential is the geometric height.

Looking at a boundary between two media

$$h_1 = \phi_1 = \phi_2 = h_2 \tag{6}$$

in every point of it.

Also if

$$q = k \frac{\partial h}{\partial \nu} \tag{7}$$

then

$$q_1 = - q_2 \tag{8}$$

2.- THE NUMERICAL PROCEDURE.

The LAPLACE equation (2) is a classical one in mathematical physics and several procedures have been devised in order to solve it.

In general a weak formulation through a set of fundations $\{\Psi_i\}_n$ i.e.a projective method is the most popular numerical approach

Taking a member Ψ_i of the family, (2) can be put as

$$-\int_D \nabla^2 \phi \,.\Psi_i = 0 \tag{9}$$

where, as before, ϕ = kh and D is the domain under study.

In order to treat with values of ϕ only in the boundary a well know limit process produces

$$- c_i \ \phi \ (x_i) + \int_{\partial D} q \,.\Psi_i = \int_{\partial D} q_i^* \ \phi \tag{10}$$

where

$$q = \partial \phi \ /\partial \nu \qquad\qquad q_i^* = \partial \Psi_i^* \ /\partial \nu$$

$$c_i = \begin{cases} 2\pi & \text{if } x_i \text{ in } D \\ \alpha & \text{if } x_i \text{ in } \partial D \\ 0 & \text{if } x_i \text{ outside } D. \end{cases}$$

where α is the vertex angle if x_i is a conical vertex of the boundary (i.e.: $\alpha = \pi$ if we are treating with a smooth surface).

A "linear element" discretization , for instance is

$$\emptyset_e = \begin{bmatrix} N_1 & \cdot & N_2 \end{bmatrix} \begin{bmatrix} \emptyset_1 \\ \emptyset_2 \end{bmatrix}$$

$$q_e = \left. \frac{\partial \emptyset}{\partial \nu} \right|_e = \begin{bmatrix} N_1 & \cdot & N_2 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \qquad (11)$$

where

$$N_1 = -\frac{1}{2} (\xi - 1) \qquad N_2 = \frac{1}{2} (\xi + 1)$$

$$-1 < \xi < 1$$

$$dsj = \frac{Lj}{2} \, d\xi$$

Finnaly a system of equations

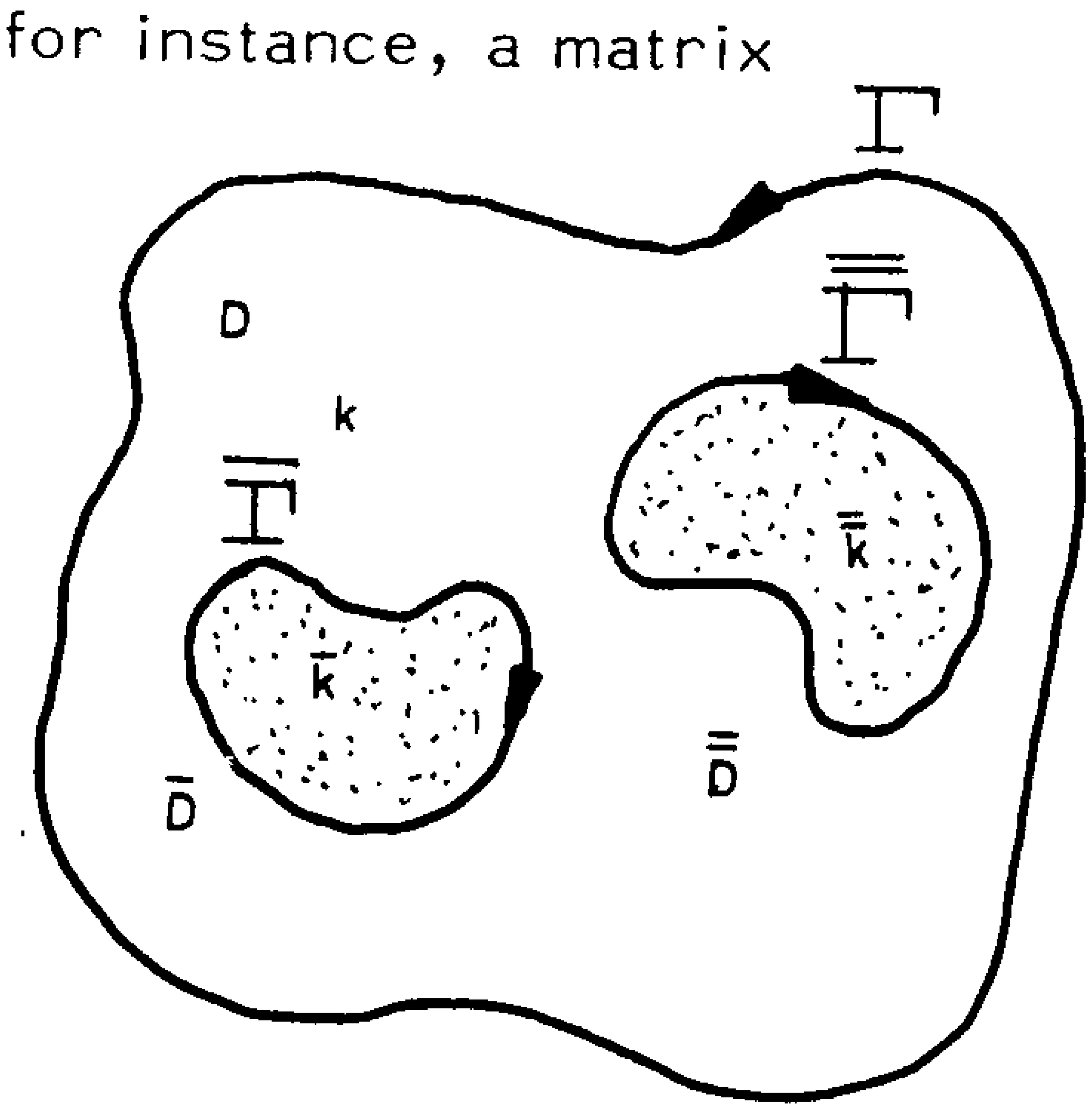
$$\underset{(n\times n)}{\underline{A}} \cdot \underset{(n\times 1)}{\underline{\emptyset}} = \underset{(n\times 2n)}{\underline{B}} \cdot \underset{(2n\times 1)}{\underline{q}} \qquad (12)$$

is prepared to receive the boundary conditions; after they are imposed we have to solve

$$\underset{(n\times n)}{\underline{K}} \cdot \underset{(n\times 1)}{\underline{x}} = \underset{(n\times 1)}{\underline{f}} \qquad (13)$$

where $\underline{X}$ receives the unknowns and $\underline{f}$ the date weighted by the apro-piate elements from $\underline{A}$ and $\underline{P}$. More details can be seen elsewhere (REF.1)

In the heterogeneous case the approach is also very simple. Assume, for instance, a matrix



D with boundary Γ and permeability k and two inclusions ($\overline{D}$, $\overline{\Gamma}$, $\overline{k}$), ($\overline{\overline{D}}$, $\overline{\overline{\Gamma}}$, $\overline{\overline{k}}$)

The field equations

$$K\nabla^2 h = 0$$
$$\overline{K}\nabla^2 h = 0 \qquad (14)$$
$$\overline{\overline{K}}\nabla^2 h = 0$$

will be established in an ordered fashion.

Assume respective discretizations with N, $\bar{n}$, $\bar{\bar{n}}$, linear elements.

Equation (12) in region D can be writen in partitioned form

$$n_t = N + \bar{n} + \bar{\bar{n}}$$

$$\underset{(n_t \times n_t)}{\begin{bmatrix} A' & \bar{A}'_1 & \bar{\bar{A}}'_1 \end{bmatrix}} \underset{(n_t \times 1)}{\begin{bmatrix} \emptyset \\ \bar{\emptyset}^1 \\ \bar{\bar{\emptyset}}1 \end{bmatrix}} = \underset{(n_t \times 2n_t)}{\begin{bmatrix} B & \bar{B}_1 & \bar{\bar{B}}_1 \end{bmatrix}} \begin{bmatrix} q \\ \bar{q}_1 \\ \bar{\bar{q}}_1 \end{bmatrix} \tag{15}$$

an similary for $\bar{D}$ $\bar{\bar{D}}$

$$\underset{(\bar{n}x\bar{n})}{\begin{bmatrix} \bar{A}'_2 \end{bmatrix}} \underset{(\bar{n}x1)}{\underline{\bar{\emptyset}}^2} = \underset{(\bar{n}x2\bar{n})}{\underline{\bar{B}}_2} \underset{(2\bar{n}x1)}{\underline{\bar{q}}^2}$$

$$\underset{(\bar{\bar{n}}x\bar{\bar{n}})}{\begin{bmatrix} \underline{\bar{\bar{A}}}'_3 \end{bmatrix}} \underset{(\bar{\bar{n}}x1)}{\underline{\bar{\bar{\emptyset}}}^3} = \underset{(\bar{\bar{n}}x2\bar{\bar{n}})}{\underline{\bar{\bar{B}}}_3} \underset{(2\bar{\bar{n}}x1)}{\underline{\bar{\bar{q}}}^3} \tag{16}$$

The conditions of (6) & (8) are

$$\begin{aligned} \bar{\emptyset}^1 &= \bar{\emptyset}^2 = \bar{\emptyset} \\ \bar{\bar{\emptyset}}^1 &= \bar{\bar{\emptyset}}^2 = \bar{\bar{\emptyset}} \\ \bar{q}1 &= -\bar{q}2 = \bar{q} \\ \bar{\bar{q}}1 &= \bar{\bar{q}}2 = \bar{\bar{q}} \end{aligned} \tag{17}$$

Grouping (15) & (16) using (17) produces

$$\begin{array}{c|c} & \begin{array}{ccc} N & \bar{n} & \bar{\bar{n}} \end{array} \\ \begin{array}{c} N+n+n \\ \bar{n} \\ \bar{\bar{n}} \end{array} & \left[\begin{array}{c|c|c} A & \bar{A}_1 & \bar{\bar{A}}_1 \\ \hline 0 & \bar{A}_2 & 0 \\ \hline 0 & 0 & \bar{\bar{A}}_3 \end{array}\right] \end{array} \left[\begin{array}{c} \emptyset \\ \hline \bar{\emptyset} \\ \hline \bar{\bar{\emptyset}} \end{array}\right] \begin{array}{c} 2(N+\bar{n}+\bar{\bar{n}}) \\ \left[\begin{array}{c|c|c} B_1 & \bar{B}_1 & \bar{\bar{B}}_1 \\ \hline 0 & -\bar{B}_2 & 0 \\ \hline 0 & 0 & \bar{\bar{B}}_3 \end{array}\right] \end{array} \left[\begin{array}{c} q \\ \hline \bar{q} \\ \bar{\bar{q}} \end{array}\right] \begin{array}{c} N+\bar{n}+\bar{\bar{n}} \\ \bar{n} \\ \bar{\bar{n}} \end{array} \tag{18}$$

Which for a Neumann problem can be written

$$
\begin{array}{c} \\ N+n+n \\ n \\ n \end{array}
\begin{array}{c}
\begin{array}{ccccc} N & \bar{n} & 2\bar{n} & \bar{\bar{n}} & 2\bar{\bar{n}} \end{array} \\
\left[\begin{array}{c|c|c|c|c}
A & \bar{A}_1 & -\bar{B}_1 & \bar{\bar{A}}_1 & -\bar{\bar{B}}_1 \\ \hline
0 & \bar{A}_2 & \bar{B}_2 & 0 & 0 \\ \hline
0 & 0 & 0 & \bar{\bar{A}}_3 & -\bar{\bar{B}}_3
\end{array}\right]
\end{array}
\begin{bmatrix} \emptyset \\ \bar{\emptyset} \\ \bar{q} \\ \bar{\bar{\emptyset}} \\ \bar{\bar{q}} \end{bmatrix}
=
\begin{array}{c} 2N \\ \begin{bmatrix} \underline{B}_1 \, \underline{q} \\ \underline{O} \\ \underline{O} \end{bmatrix} \end{array}
\begin{array}{c} N+n+n \\ \bar{n} \\ \bar{\bar{n}} \end{array}
\qquad (19)
$$

Imposing conditions on $\bar{q}$ & $\bar{\bar{q}}$, for instance

$$
\bar{q}^{b} = \bar{q}^{a} \quad \text{or} \quad \bar{\bar{q}}^{b} = \bar{\bar{q}}^{a} \qquad (20)
$$

in case of smooth boundaries. One can solve as system of $N+2\,(\bar{n} + \bar{\bar{n}})$ equations with $N + (\bar{n} + \bar{\bar{n}})$ unknowns i.e.: the potentials of the other boundary and the $2(\bar{n}+\bar{\bar{n}})$ potentials and fluxes at the interior boundaries.

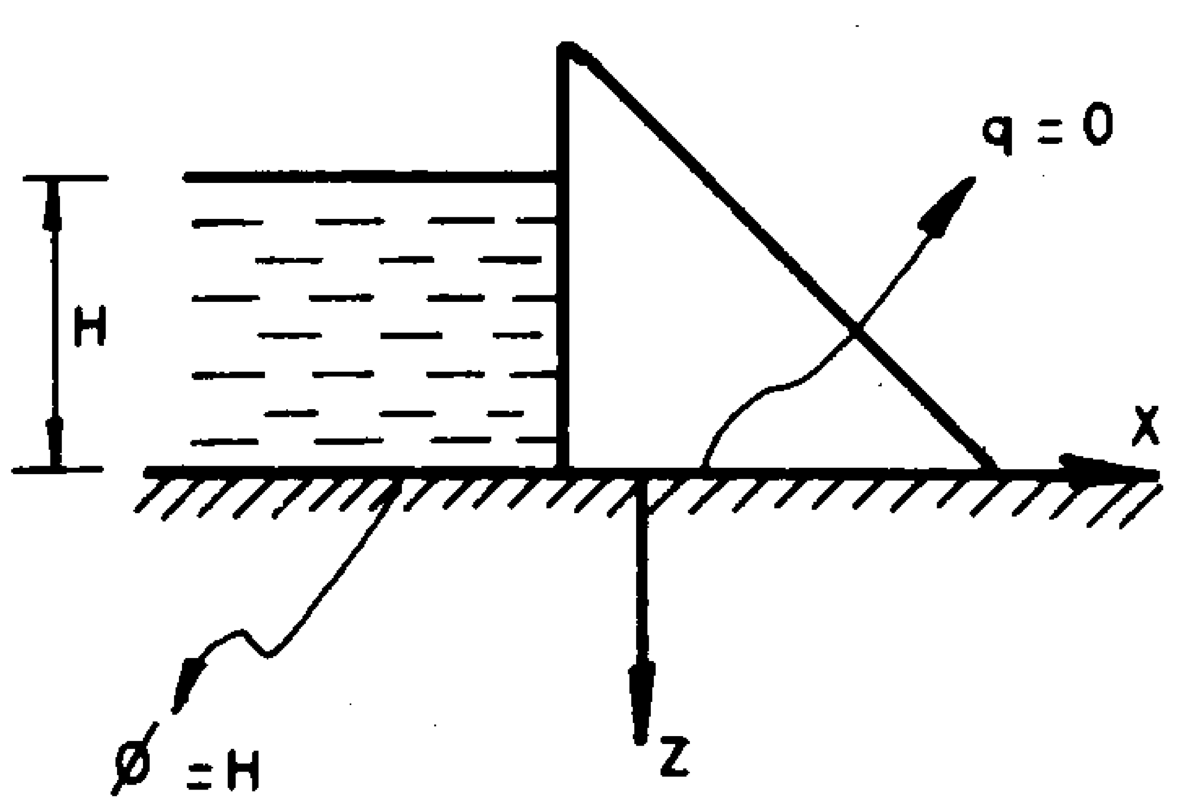
3.- EXAMPLES

In order to show the capabilities of the method we present some -- exemples.

The first one (plate 1,2,3) treats a classical problem. The filtration under a dam within a homogeneous media.

In order to check the accuracy the results have been compared with the analytical solution for a halfspace.

$$
\emptyset = \frac{k\ H}{\pi} \quad \text{arc cos} \quad (x/b)
$$



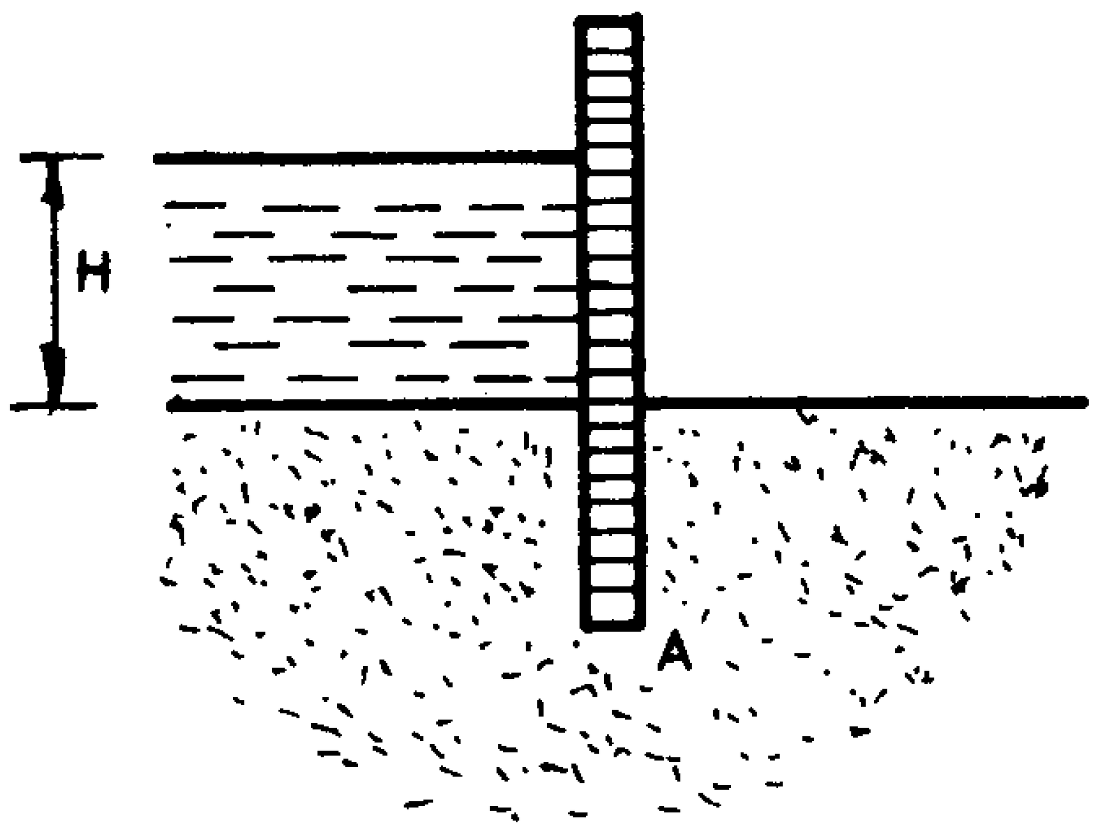
The registered diferences being clearly produced by the finite - dimensions of the studied problem.

It is interesting to see how only the boundary has to be discretized and the good convegence of results with increasing meshes.

The second exemple, (plate 4) the classical sheet -wall, is inte- - resting because the singularity which appears at the end of the sheet- -wall.

The best to establish the conditions is to solve a symetric and a - skewsymetric problem and then to superpose bot h .

The symetric case is symply a constant function and the skewsymetric one has to be carefully treated in order to manege the singularity at A.



As it is well know this is due to the presence of a $r^{-\frac{1}{2}}$ factor in the -- series expansion round A.

To correct the results one can draw a $r^{-\frac{1}{2}}$ $\emptyset$ graph and extrapolate the results to get the corresponding - "concentration factor".

Some authors use to prepare a singular element, for exemple, the 1/4 parabolic one in order to simulate automacally singularity.

As can be seen on plate 2, results compare well with the halfspace solution.

The final exemple is the study of the flow through a infinite band with aperiodic array of circular inclusions.

Using symetry one needs only to discretize a square- shaped area with a hole in it.

In plate 5, we present the evolution of potentials round a quarter of the circular inclusion, as a function of the ratio of permeabilities. Note than in the case of impervious inclusion the results are of the classical problem of perfect fluid flow round a cylinder in the middle of a rectangular pipe.

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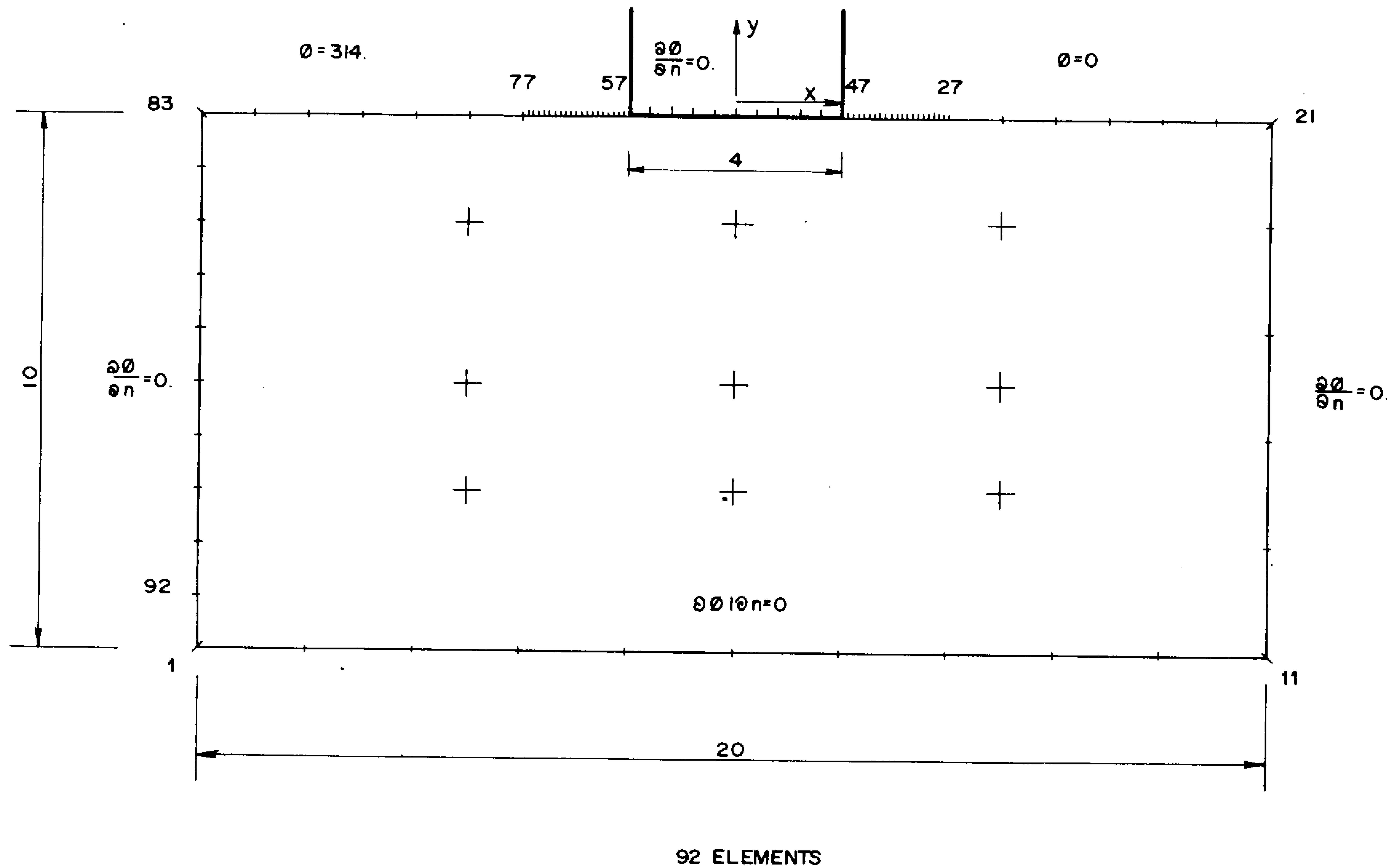
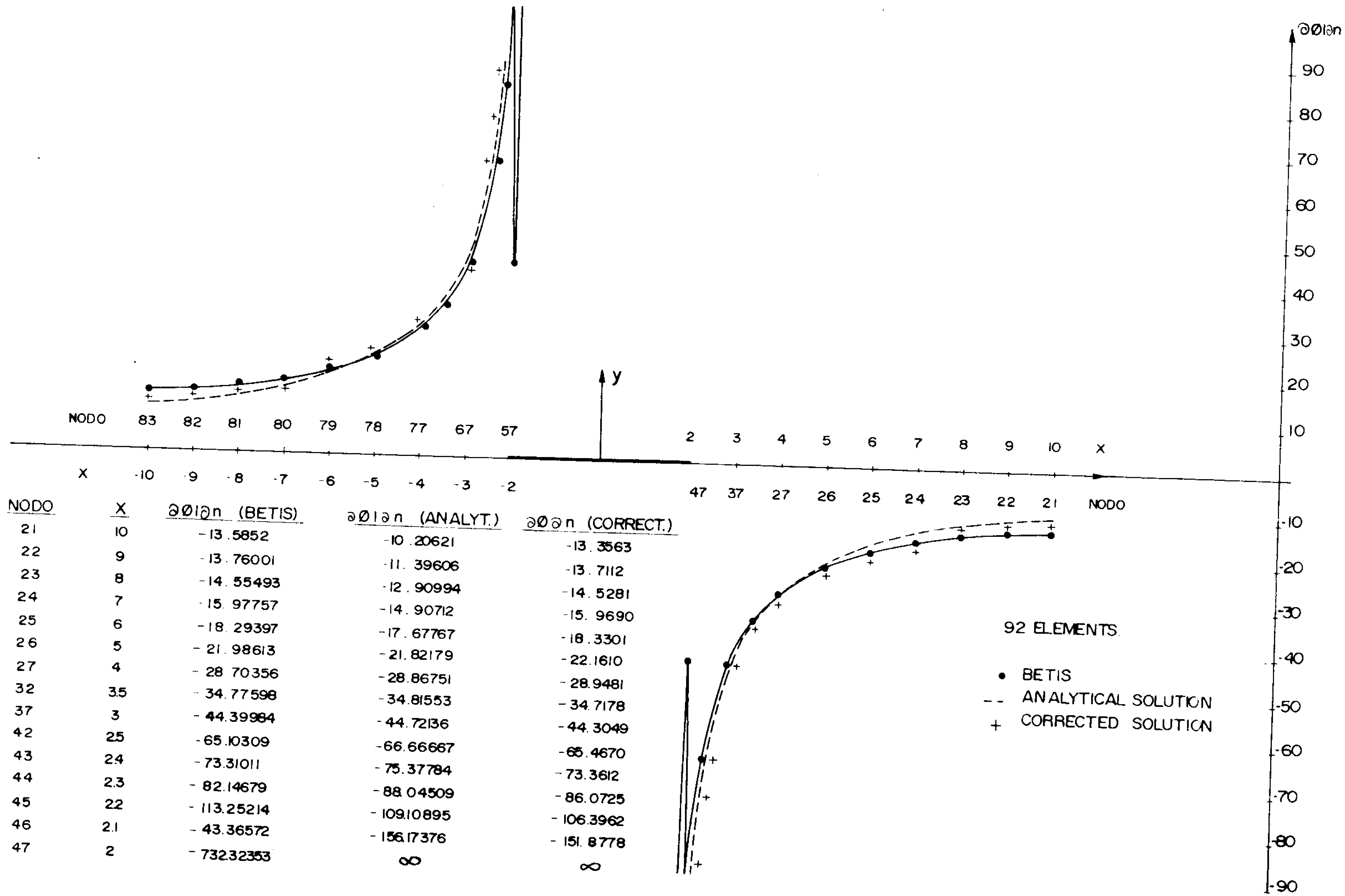


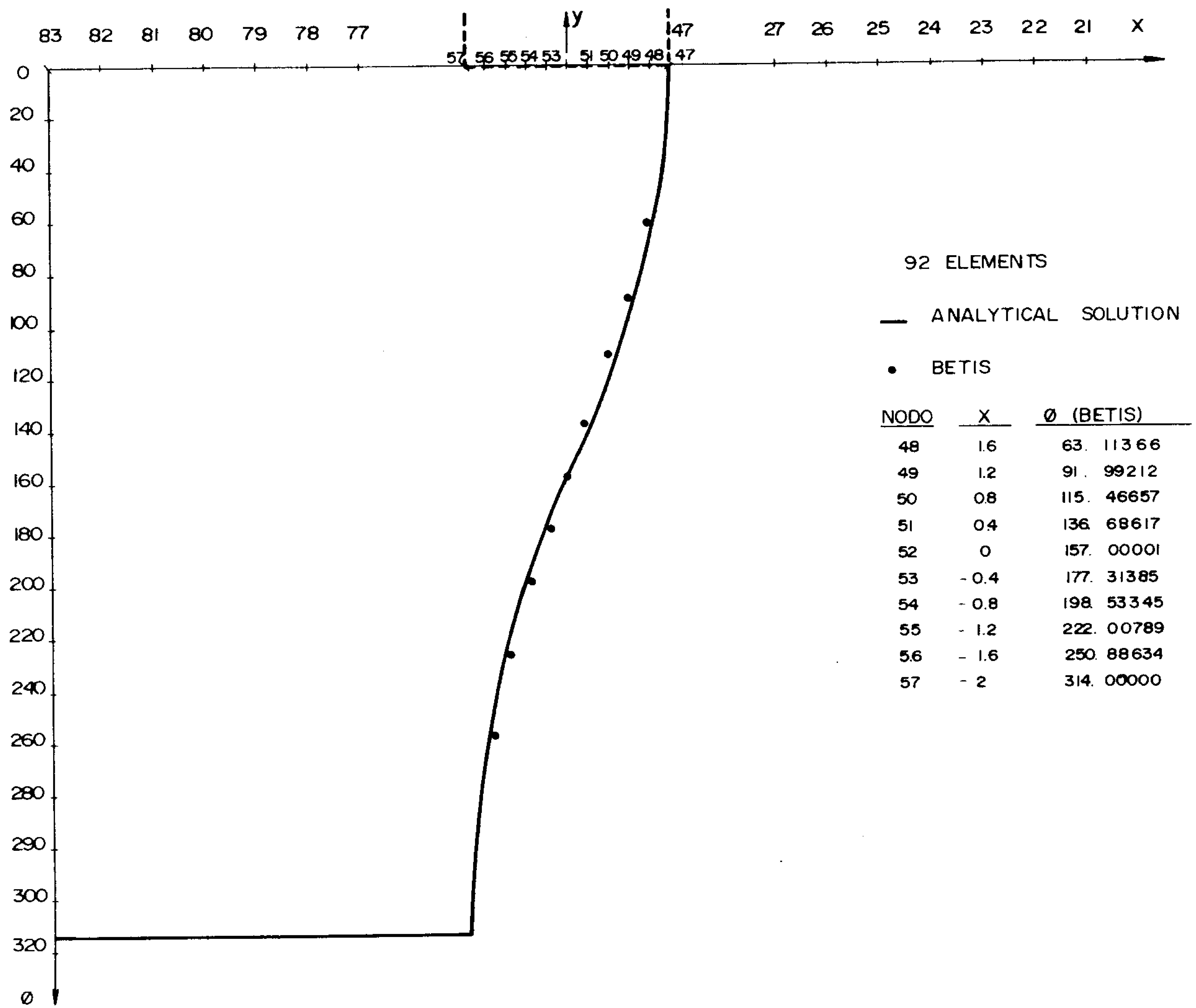
Ø = 314.
77
57
$\frac{\partial Ø}{\partial n}$ = 0.
y
x
47
27
Ø = 0
83
21
4
10
$\frac{\partial Ø}{\partial n}$ = 0.
$\frac{\partial Ø}{\partial n}$ = 0.
92
∂Ø/∂n=0
1
11
20
92 ELEMENTS

Plate 1



NODO	X	∂Ø/∂n (BETIS)	∂Ø/∂n (ANALYT.)	∂Ø/∂n (CORRECT.)
21	10	-13.5852	-10.20621	-13.3563
22	9	-13.76001	-11.39606	-13.7112
23	8	-14.55493	-12.90994	-14.5281
24	7	-15.97757	-14.90712	-15.9690
25	6	-18.29397	-17.67767	-18.3301
26	5	-21.98613	-21.82179	-22.1610
27	4	-28.70356	-28.86751	-28.9481
32	3.5	-34.77598	-34.81553	-34.7178
37	3	-44.39984	-44.72136	-44.3049
42	2.5	-65.10309	-66.66667	-65.4670
43	2.4	-73.31011	-75.37784	-73.3612
44	2.3	-82.14679	-88.04509	-86.0725
45	2.2	-113.25214	-109.10895	-106.3962
46	2.1	-43.36572	-156.17376	-151.8778
47	2	-732.32353	∞	∞

Plate 2



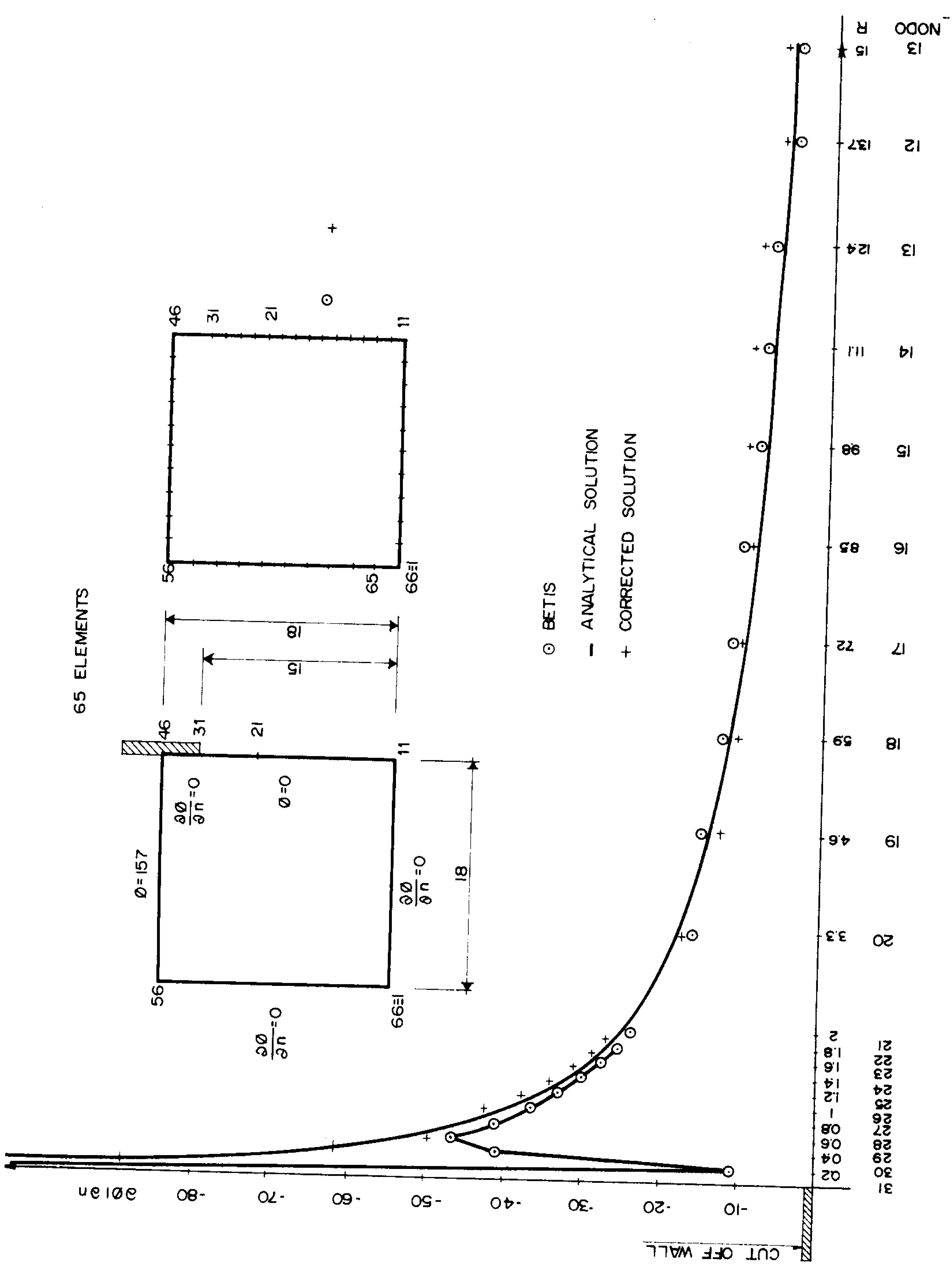
92 ELEMENTS

— ANALYTICAL SOLUTION

• BETIS

NODO	X	Ø (BETIS)	Ø (AMALYTICAL)
48	1.6	63. 11366	64.35011
49	1.2	91. 99212	92.72952
50	0.8	115. 46657	115.92795
51	0.4	136. 68617	136.94384
52	0	157. 00001	157.00000
53	- 0.4	177. 31385	177.21542
54	- 0.8	198. 53345	198.23132
55	- 1.2	222. 00789	221.42974
5.6	- 1.6	250. 88634	249.80915
57	- 2	314. 00000	314.15927

Plate 3


65 ELEMENTS
⊙ BETIS
— ANALYTICAL SOLUTION
+ CORRECTED SOLUTION
CUT OFF WALL
NODO
R
Ø=157
Ø=0

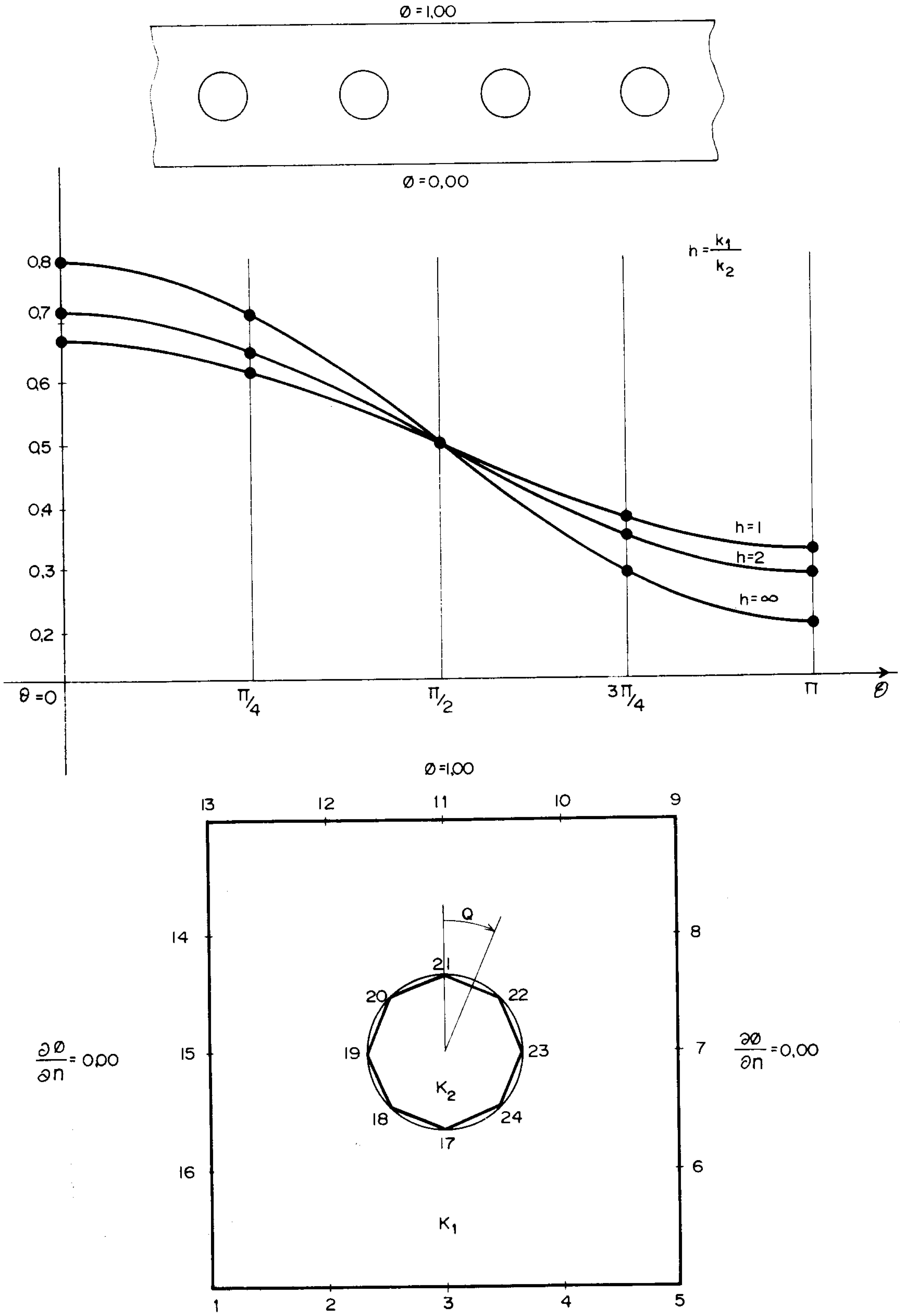

Ø = 1,00
Ø = 0,00
0,8
0,7
0,6
0,5
0,4
0,3
0,2
$h = \frac{k_1}{k_2}$
h = 1
h = 2
h = ∞
θ = 0
π/4
π/2
3π/4
π
θ
Ø = 1,00
13 12 11 10 9
14 8
15 7
16 6
1 2 3 4 5
21 22 23 24 17 18 19 20
Q
K_2
K_1
$\frac{\partial Ø}{\partial n} = 0,00$
$\frac{\partial Ø}{\partial n} = 0,00$
Ø = 0

Plate 5